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Wave Calculus Based Upon Wave Logic

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A number operator has been introduced based upon the binary (p-nary) presentation of numbers. This operator acts upon a numerical state vector. Generally the numerical state vector describes numbers that are not precise but smeared in a quantum sense. These states are interpreted in wave logic terms, according to which concepts may exist within the inner language of a phenomenon that in principle cannot be translated into the language of the investigator. In particular, states may exist where mean values of a quantity, continuous in classical limits, take only discrete values. Operators for differentiation and integration of operator functions are defined, which take the usual form in the classical limit.

1. WAVE LOGIC AND ITS APPLICABILITY

Suppose that a quantity X' is measured with a precision of only $\Delta X' \geqslant \Lambda$, independent of the experimental procedure. Let Λ be called the minimal length, and then let us measure X' in units of Λ by introducing

$$X = X'/\Lambda \tag{1.1}$$

Obviously, only numbers $X \gg 1$ have sufficiently clear meaning. This cannot be said for numbers $X < 1$. In defining such numbers we are compelled to employ as a base not the real experiment for measuring X' but only analogies and images. It is important, however, that there always exist external objects (independent of the quantity X') which permit the observer to construct the classical scale of numbers ordinarily employed, to be denoted as X_0. The observer is also compelled to employ classical logic to describe phenomena, and in a certain sense which will become clear below it may be said that the use of the classical scale of numbers is related to the use of classical logic.

However, there is no guarantee that the inner logic of the phenomenon connected with X' coincides with the classical logic of the investigator. Concepts and symbols that have a definite meaning within this inner logic may not have a unique interpretation within the logic of the investigator and vice versa. In such cases the corresponding quantities cannot be measured by the investigator with sufficient precision.

When such a situation arises it may be said with equal fairness that X' cannot be precisely measured because it is impossible to give it an adequate definition, and thus the numbers X (and propositions based upon them) have no unique meaning; or that it is impossible to give X' an adequate definition because X' in principle cannot be precisely measured.

Wave logic as described in Orlov (1975)[1] is a convenient tool for describing such situations (of course, from the point of view of one side, in this case from the point of view of the observer). To be more exact, in terms of this tool, it is not a matter of different logics but of different languages within the framework of one and the same wave logic, or of different points of view. In such a case the language of the observer and the inner language of the phenomenon have no exact mutual translation "because" the points of view are incompatible.

Some real situations may be even more complicated. That is, certain conceptions can have no exact definitions from *any* point of view. The apparatus of wave logic also permits a description of such a case, by going from state vectors to density matrices (see below). We continually encounter such complicated situations in the humanities and when communicating with one another.

Wave logic is a natural generalization of the classical two-valued logics. In so far as within any nonequivalent proposition no information exists concerning its validity or falsity, the idea suggests itself of representing the proposition as an operator acting on some function in which such information is included. A simple generalization of such functions leads to the SU_2 rotation group in the "truth space." From the hypothesis of the existence of the SU_2 group follows the existence of multiple-valued wave logic. Below we relate the p-nary presentation of a number and the application of p-nary wave logic. But the main results appear to be independent of the dimension p of the logic.

The tool for analyzing nonprecise numbers developed in this paper could be considered independently, without wave-logical interpretation. The wave-logical interpretation may reflect the train of thought of the author that led to these results with independent meaning.

[1] Only Part I of this paper has been published.

2. THE NUMBER OPERATOR

The binary presentation of any number X is

$$X = s \sum_{k=-\infty}^{\infty} \lambda_k / 2^k \tag{2.1}$$

where $s = \pm 1$ designates the sign of the number, and $\lambda_k = 0, 1$.

Let us relate this presentation to an infinite row of cells numbered k with all integer values from $-\infty$ to ∞. We shall say that cell k is filled if $\lambda_k = 1$ and void if $\lambda_k = 0$. A special sign cell is filled if $s = +1$ and void if $s = -1$.

We now introduce an enumerable infinite set of atomic propositions denoted as

$$\tfrac{1}{2}\sigma_{3k} \equiv \text{cell number } k \text{ is filled} \tag{2.2}$$

In accordance with (1.1) σ_{3k} is the 2×2 operator affecting the kth cell:

$$\sigma_{3k} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_k \tag{2.3}$$

Information about the validity or falsity of proposition (2.2) is contained in the kth wave function—the spinor φ_k. In the two limiting cases corresponding to classical logic, namely, when proposition (2.2) is precisely true or precisely false ($\lambda_k = 1$ or 0) we have

$$(\varphi_k)_{\text{class}} \equiv \varphi_0^{\lambda_k}, \qquad \varphi_0{}^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \varphi_0{}^0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.4}$$

$$\sigma_{3k}\varphi_0^{\lambda_k} = (2\lambda_k - 1)\varphi_0^{\lambda_k} \tag{2.5}$$

Let us now redefine (2.1) by introducing the number *operator* x:

$$x \equiv s \sum_{k=-\infty}^{\infty} (\tfrac{1}{2} + \tfrac{1}{2}\sigma_{3k})/2^k \tag{2.6}$$

This operator must act on the *numerical state vector* written as the tensor product of all the kth wave functions. In the classical case (i.e., when $\lambda_k = 0$ or 1) the state vector equals

$$F_0^{(x_0)} = S \prod_{k=-\infty}^{\infty} \varphi_0^{\lambda_k}, \qquad sS = \pm S, \qquad \sigma_{3k'} F_0^{(x_0)} = \sigma_{3k'}\varphi_0^{\lambda_{k'}} \prod_{k \neq k'} \varphi_0^{\lambda_k} \tag{2.7}$$

Here S is a sign function containing information about the sign of the number x_0 which is completely determined by the sequences of numbers λ_k by using formula (2.1). Thus the information on the number is not contained in presentation (2.6), but it is transferred to the state vector. The number x_0 itself results from the action of the operator x upon the state vector:

$$xF_0^{(x_0)} = x_0 F^{(x_0)}, \qquad x_0 = s \sum_{k=-\infty}^{\infty} \lambda_k/2^k \tag{2.8}$$

There is a certain ambiguity in the presentation of binary-rational numbers by formula (2.1). Let us agree upon the presentation of binary-rational numbers by a finite amount of quantities $\lambda_k = 0, 1$. Thus a mutually unique correspondence between numbers x_0 and sequences λ_k is established. Thereafter we may define the scalar product of state vectors by

$$(F_0^{(x_0')}, F_0^{(x_0)}) \equiv F_0^{(x_0')\dagger} F_0^{(x_0)} = \prod_{k=-\infty}^{\infty} \delta_{\lambda_k' \lambda_k} \delta_{s's} \tag{2.9}$$

where δ_{ab} is the Kronecker symbol and $\dagger$ is the Hermitian conjugate.

Any arbitrary state vector may be represented by a (nondenumerable) set of orthonormal vectors (2.7).

The average value of an arbitrary operator in some state is defined in the usual manner. In particular

$$\begin{aligned} \langle x \rangle &= (F_0^{(x_0)}, xF_0^{(x_0)}) = s \sum_{k=-\infty}^{\infty} \frac{1}{2^k} \left(\varphi_0^{\lambda_k}, \frac{1+\sigma_{3k}}{2} \varphi_0^{\lambda_k} \right) \\ &= s \sum \lambda_k/2^k = x_0 \end{aligned} \tag{2.10}$$

In the vector representation of $F_0^{(x_0)}$ all the operators σ_{3k} and together with them the operator x are diagonal.

A similar tool may be developed for any p-nary representation of a number x,

$$x = s \sum_{k=-\infty}^{\infty} \left(\frac{p-1}{2} + \Omega_{3k} \right) \Big/ p^k \tag{2.11}$$

where Ω_{3k} is the operator of proposition (2.2) in the p-nary logic. Its (diagonal) elements are

$$\Omega_{3k} = \frac{p-1}{2}, \frac{p-3}{2}, \ldots, -\frac{p-1}{2}$$

Ω_{3k} acts upon the p-dimensional wave function (column).

3. REAL SYSTEM STATE VECTOR

In general the numerical state vector F may not coincide with (2.7); moreover, by our initial assumptions on the properties of measurements of the quantity X', the numerical state vector of this quantity is sure to differ from any of the $F_0^{(x_0)}$. We assume that in such a case the independence of propositions (2.2) referring to different cells is valid, so that

$$F = S \prod_{k=-\infty}^{\infty} \varphi_k \tag{3.1}$$

where spinors φ_k are of the form

$$\varphi_k = \begin{pmatrix} \alpha_k \\ \beta_k \end{pmatrix}, \qquad |\alpha_k|^2 + |\beta_k|^2 = 1 \tag{3.2}$$

and $|\alpha_k|^2$ is the probability of proposition (2.2) being true, while $|\beta_k|^2$ is the probability of it being false.

Expanding F in the orthonormalized basis

$$F = \sum_{x_0} a(x_0) F_0^{(x_0)} \tag{3.3}$$

we may assert that $|a(x_0)|^2$ is the probability that a single measurement of the quantity x will give x_0. As x_0 runs over a continuum of values it is better to speak about the density of the probability and to compare probabilities relative to one another.

Let us consider physical (or other) situations where

$$|\alpha_k|^2 = \begin{cases} 1 \text{ or } 0 & \text{for } k \leqslant 0 \\ \frac{1}{2} & \text{for } k \geqslant 1 \end{cases} \tag{3.4}$$

for instance, $\alpha_k = 2^{-1/2}$, $\beta_k = \pm 2^{-1/2}$. The set of state vectors of this type has the same cardinality as the set of vectors $F_0^{(x_0)}$. We shall write the numerical state vector of this type in the form

$$F_{1/2} = S \prod_{k=-\infty}^{0} \varphi_0^{\lambda_k} \prod_{k=1}^{\infty} (\alpha_k \varphi_{0k}^1 + \beta_k \varphi_{0k}^0) \tag{3.5}$$

The first factors in this product describe the integer part of the number $N = s \sum \lambda_k / 2^k$, which is given precisely. The second factors describe the fractional part of the number and it is easy to see that all the points between $|N|$ and $|N + 1|$ are equivalent. Thus it should be expected that the average value of the quantity x in any of these states should be $\langle x \rangle_{1/2} = s(N + \frac{1}{2})$.

This result is independent of the dimension p of the logic applied. If the square of the modulus of all the coefficients of the p-dimensional wave function are equal, $|\alpha|^2 = 1/p$, then $\langle \Omega_{3k} \rangle = 0$ and

$$\langle x \rangle_{1/2} = \bar{\bar{S}}(N + \tfrac{1}{2}), \qquad \langle x' \rangle_{1/2} = \bar{\bar{s}}(N + \tfrac{1}{2}) \cdot \Lambda \tag{3.6}$$

Let us consider one more possible situation. Let the inner language of phenomena related to the quantity x' be "rotated" relative to the classical language of the observer in such a manner that for the real state

$$\varphi_k^{(\lambda_k)} = O_k \varphi_{0k}^{\lambda_k} = \cos\beta_k \varphi_{0k}^{\lambda_k} + (2\lambda_k - 1)\sin\beta_k \varphi_{0k}^{1-\lambda_k} \tag{3.7}$$

where the rotation operator in the kth cell is

$$O_k = \begin{pmatrix} \cos\beta_k & -\sin\beta_k \\ \sin\beta_k & \cos\beta_k \end{pmatrix} \tag{3.8}$$

In this case, as can easily be checked,

$$\langle x\rangle^{\{\lambda_k\}} = x^{\{\lambda_k\}} - s\sum_{k=-\infty}^{\infty} \frac{(2\lambda_k - 1)\sin^2\beta_k}{2^k} \tag{3.9}$$

where $\{\lambda_k\}$ is that sequence that corresponded to the "nonrotated" state

$$F^{\{\lambda_k\}} = \prod_{k=-\infty}^{\infty} O_k F_0^{(x_0)}, \qquad x_0 \equiv x_0^{\{\lambda_k\}} = \bar{\bar{s}} \sum_k \lambda_k/2^k \tag{3.10}$$

If $\sin^2\beta_k = 0$ for $k \leqslant 0$ and $\sin^2\beta_k = \frac{1}{2}$ for $k \geqslant 1$, (3.9) will again give the result of (3.6). The situation may be such that $\sin^2\beta_k$ gradually increases from zero to $\frac{1}{2}$ at $k \geqslant 1$. This corresponds to a gradual transition from complete clearness to complete unclearness about the truth of the proposition (2.2). It may be said that each division by 2, beginning with the division of unity, gives its specific contribution to the spread (smearing) of the resulting numbers. Let us for instance assume that

$$\sin^2\beta_k = \begin{cases} 0, & k \leqslant 0 \\ \frac{1}{2}(1 - e^{-\alpha k}), & k \geqslant 1 \end{cases} \tag{3.11}$$

In this case the measurement results for the averages $\langle x\rangle$ are not located only at points of the discrete grid (3.6); also permissible are $\langle x\rangle$ lying within certain segments of the intervals between integer numbers. Although single measurements may produce any values for the quantity x, for average values the regions in the vicinity of integers $N + \alpha$ are prohibited, where $1/\alpha$ is (in order of magnitude) that cell number beginning with which a strong uncertainty exists in the truth of (2.2). This result including the limiting case (3.6) gives hope that the tool for "wave calculus" proposed here may become useful in describing nonlocality in the theory of elementary particles, if such nonlocality exists.

It should be emphasized that the location of prohibited points is independent of the choice of the origin and of the direction of the coordinate axis in space if the space is not unidimensional. It is essential however that the coordinate system is chosen *before* verification of predictions concerning the mathematical expectation of quantities, i.e., before making measurements. This situation is analogous to that which exists when measuring the spin of a particle when the effect of the measuring device determining "the z axis" leads to the appearance of the discrete spectrum of spin projections upon this axis. Quantum mechanics does not describe the "objective world" but what we can observe with the assistance of measuring devices when their influence upon the object is the minimum possible in principle. As for the situation being described in this paper, it should be especially emphasized at this point that not only the act of making the measurement but the choice of the language itself with which the investigator intends to describe the results of measurements is a violation of the object, which perhaps has its own language, in principle noninterpretable into the language of the investigator, no matter what language is chosen by the investigator. In particular, the conception "the coordinate origin" cannot be exactly translated into "the language of the object" if situations of types (3.4) or (3.11) exist.

The following elucidates the meaning of these statements. The vectors

$$F^{\{\lambda_k\}} = S \prod_{k=-\infty}^{\infty} [\cos \beta_k \varphi_{0k}^{\lambda_k} + (2\lambda_k - 1) \sin \beta_k \varphi_{0k}^{1-2\lambda_k}] \tag{3.12}$$

comprise a complete orthonormal set. Propositions represented by this set

$$\bar{\sigma}_{3k} = O\sigma_{3k}O_k^{-1} \tag{3.13}$$

have a unique meaning and the number operator

$$\bar{x} = \left(\prod_k O_k\right) x \left(\prod_k O_k^{-1}\right) = \bar{\bar{s}} \sum_{k=-\infty}^{\infty} \tfrac{1}{2}(1 + \bar{\sigma}_{3k})/2^k \tag{3.14}$$

is diagonal in this representation:

$$\bar{x}F^{\{\lambda_k\}} = s\left(\sum_{k=-\infty}^{\infty} \lambda_k/2^k\right) F^{\{\lambda_k\}}, \qquad F^{\{\lambda_k\}} \equiv F^{(\bar{x})} \tag{3.15}$$

The formal expression of the impossibility in principle of a unique correspondence between the language of the investigator and the inner language of the phenomenon is the noncommutativity of the operators σ_{3k} and $\bar{\sigma}_{3k}$ and consequently of the operators x and $\bar{x}$:

$$x\bar{x} - \bar{x}x = is \sum_{k=-\infty}^{\infty} \frac{(\sin 2\beta_k)\sigma_{2k}}{2^k}, \qquad \sigma_{2k} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}_k \tag{3.16}$$

The mean $\langle\sigma_{2k}\rangle^{\lambda_k}$ in state (3.12) equals zero, and as $\sigma_{2k}^2 = 1$ then

$$\langle x\bar{x} - \bar{x}x\rangle^{\{\lambda_k\}} = -\sum_k \sin^2 2\beta_k/4^{2k+1} \tag{3.17}$$

In the case of (3.4) when $\sin^2\beta_k = \cos^2\beta_k = \frac{1}{2}$, $k \geqslant 1$, we obtain

$$|\langle x\bar{x} - \bar{x}x\rangle^{\{\lambda_k\}}|^{1/2} = \frac{1}{2\sqrt{7}} \tag{3.18}$$

Such is the order of magnitude of uncertainty of the concept of a number resulting from the inherent inadequacy of the language of the observer (Orlov, 1974).[2]

The language interpretation of this effect, however, cannot be considered strictly substantiated, insofar as the investigator deals with instrument readings and does not seem able to distinguish "language" effects from instrument effects upon the object's state. One general argument in favor of the language interpretation is that the effect of the act of measuring on the state of a system "object + instrument" is described by quantum mechanics, which is an extremely restricted theory. To explain effects beyond the limits of this theory (if they really exist) one should draw on new principles. The possible influence of the language on the state of the system "object + instrument + description system" is a new principle (at the same time lying in the channel of quantum mechanical philosophy).

The situation may be such that no unique interpretation exists within the inner language of the phenomenon of the proposition $\bar{\sigma}_{3k}$. In such a case the state vectors (3.13) should be substituted by some density matrices.

4. FUNCTION OPERATORS

In the representation where operator x is diagonal the operator function $\phi(x)$ is determined by the following relation:

$$\phi(x)F^{(x_0)} = \phi(x_0)F_0^{(x_0)} \tag{4.1}$$

So, if F is determined by (3.3), then

$$\phi(x)F = \sum_{x_0} a(x_0)\phi(x_0)F_0^{(x_0)} \tag{4.2}$$

The Taylor series expansion of $\phi(x)$ may be convenient in practice:

$$\phi(x) = \phi(\langle x\rangle) + \phi'(\langle x\rangle)\Delta x + \tfrac{1}{2}\phi''(\langle x\rangle)\Delta x^2 + \cdots, \qquad \Delta x \equiv x - \langle x\rangle \tag{4.3}$$

[2] In Orlov (1974) expressions of type (3.16) were postulated.

where $\phi(\langle x \rangle)$ is the usual (numerical) function of $\langle x \rangle$. The mathematical expectations and matrix elements for some of the first few moments in situations described by (3.7) are

$$\langle \Delta x^2 \rangle = \sum_k \frac{\sin^2 \beta_k \cos^2 \beta_k}{4k} \tag{4.4}$$

$$= \tfrac{1}{12} \qquad \text{for } \sin^2 \beta_k = \cos^2 \beta_k = \tfrac{1}{2}, \qquad k \geqslant 1 \tag{4.5}$$

$$\langle \Delta x^3 \rangle = s \sum_k (2\lambda_k - 1) \sin^2 \beta_k \cos^2 \beta_k \cos 2\beta_k / 8^k \tag{4.6}$$

$$= 0 \text{ for the case (4.5)} \tag{4.7}$$

$$\langle \Delta x^4 \rangle = \sum_k \frac{1}{4^k} \left(\sin^2 \beta_k \cos^2 \beta_k \cos 2\beta_k + \frac{1}{4^k} \sin^4 \beta_k \cos^4 \beta_k \right)$$

$$+ 5 \sum_{k \neq l} \sum \frac{1}{4^{k+l}} \sin^2 \beta_k \cos^2 \beta_k \sin^2 \beta_l \cos^2 \beta_l \tag{4.8}$$

In the particular case when x is an operator and p a common number, then

$$\langle e^{ipx} \rangle = e^{ip\langle x \rangle} \left(1 - \frac{p^2}{24} + \frac{13p^4}{16 \cdot 24 \cdot 45} - \cdots \right) \tag{4.9}$$

$$= f(p) \exp [ips(N + \tfrac{1}{2})]$$

where $f(p)$ is the usual numerical function of p.

Note that dispersion (4.5) is independent of the dimension p of the logic applied just as is true of $\langle x \rangle$. Indeed, in the case of equal probability for any answer to the question whether or not proposition Ω_{3k} is true we have

$$\langle \Omega_{3k} \rangle = 0$$

and

$$\langle \Omega_{3k}^2 \rangle = \langle \Omega_{2k}^2 \rangle = \frac{1}{3} \frac{p-1}{2} \frac{p+1}{2} = \frac{p^2 - 1}{12}$$

Thus

$$\langle x^2 \rangle = \langle x \rangle^2 + \frac{p^2 - 1}{12} \sum_{k=1}^{\infty} \frac{1}{p^{2k}} = (N + \tfrac{1}{2})^2 + \tfrac{1}{12} \tag{4.10}$$

Let us further write a few matrix elements [for case (2.7)]. Since

$$xF^{\{\lambda_i\}} = x_0F^{\{\lambda_i\}} - s\sum_{k=-\infty}^{\infty}\frac{\sin\beta_k}{2^k}\varphi_{0k}^{1-\lambda_k}$$
$$\times\prod_{n\neq k}(\cos\beta_n\varphi_{0n}^{\lambda_n} + (2\lambda_n - 1)\sin\beta_n\varphi_{0n}^{1-\lambda_n}) \tag{4.11}$$

operator x in addition to diagonal elements contains only matrix elements for transitions $\lambda_k \to 1 - \lambda_k$, $\lambda_i \to \lambda_i$, $i \neq k$, where k is one (and only one) of the cell numbers:

$$\langle 1 - \lambda_k|x|\lambda_k\rangle \equiv (F^{\{\lambda_i;1-\lambda_k\}}, xF^{\{\lambda_i;\lambda_k\}}) = -s\frac{\sin\beta_k\cos\beta_k}{2^k} \tag{4.12}$$

(in the braces { ; } the varying λ_k are written separately from nonvarying λ_i). Hence

$$\langle 1 - \lambda_k|x^2|\lambda_k\rangle = -s\frac{\sin\beta_k\cos\beta_k}{2^k}\left[2\langle x\rangle^{\{\lambda_k\}} + s\frac{(1-2\lambda_k)\cos 2\beta_k}{2^k}\right] \tag{4.13}$$

$$\langle 1 - \lambda_k; 1 - \lambda_n|x^2|\lambda_k; \lambda_n\rangle = \frac{\sin 2\beta_k \sin 2\beta_n}{2^{k+n+1}} \tag{4.14}$$

5. DIFFERENTIATION AND INTEGRATION

Let us define an operator of variation of operator x by

$$d_n x = [x\sigma_{1n} - \sigma_{1n}x] = is\sigma_{2n}/2^n \tag{5.1}$$

where

$$\sigma_{1n} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_n, \qquad [\sigma_3\sigma_1 - \sigma_1\sigma_3]_n = i\sigma_{2n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_n \tag{5.2}$$

The meaning of definition (5.1) is as follows. The operator σ_{1n} performs the substitution $\lambda_n \to 1 - \lambda_n$ within the nth cell of the state $F_0^{(x_0)}$. Thus if the operator x acts after the operator σ_{1n} then the result of its action changes by $\pm\frac{1}{2}^n$ as compared to the inverse order of action of the operators:

$$d_n x F_0^{(x_0)} = \lim_{n\to\infty}\frac{s(1-2\lambda_n)}{2^n}F_0^{[x_0+s(1-2\lambda_n)/2^n]} \tag{5.3}$$

Similarly we shall define an operator of variation of a function $\phi(x)$ as

$$d_n\phi(x) = [\phi(x)\sigma_{1n} - \sigma_{1n}\phi(x)] \tag{5.4}$$

Now the derivative operator $\phi' = d\phi/dx$ is determined from the equation

$$\lim_{n\to\infty} [d_n\phi - \phi' \, d_n x] = 0 \tag{5.5}$$

or

$$\phi' = -\lim_{n\to\infty} is\sigma_{2n}2^n[\phi\sigma_{1n} - \sigma_{1n}\phi] \tag{5.6}$$

Let us find, for example, the operator dx^2/dx. $x^2\sigma_{1n} - \sigma_{1n}x^2 = \{x, [x, \sigma_{1n}]\}$. It can be easily seen that in the limit $n \to \infty$ both terms commute:

$$[x(x\sigma_{1n} - \sigma_{1n}x) - (x\sigma_{1n} - \sigma_{1n}x)x] = \sigma_{1n}/2^{2n} \tag{5.7}$$

Thus $dx^2/dx = 2x$. In general

$$dx^n/dx = nx^{n-1} \tag{5.8}$$

In particular, for $n = -1$ it follows that

$$\lim [x^{-1}\sigma_{1n} - \sigma_{1n}x^{-1} + x^{-2}(x\sigma_{1n} - \sigma_{1n}x)] = 0$$

from the identity

$$\lim_{n\to\infty} \{2x\sigma_{1n}x - x^2\sigma_{1n} - \sigma_{1n}x^2\} = 0$$

(the brackets have an order of magnitude equal to $\frac{1}{2}^{2n}$).

Various possibilities for integration exist in this theory.

If function $\phi(x)$ has an antiderivative so that $\phi(x) = d\Phi/dx$ then an integral can be determined by

$$\int_a^x \phi(x)\, dx = \Phi(x) - \Phi(a) \tag{5.9}$$

Let us consider as an example the mean value of the integral of x^2:

$$\left\langle \int_a^x x^2\, dx \right\rangle = \tfrac{1}{3}(\langle x^3\rangle - a^3) = \tfrac{1}{3}(\langle x\rangle^3 - a^3) + \langle x\rangle(\langle x^2\rangle - \langle x\rangle^2)$$

For the case (3.4) this integral becomes

$$\tfrac{1}{3}[(N + \tfrac{1}{2})^3 - a^3] + \tfrac{1}{12}(N + \tfrac{1}{2}) \tag{5.10}$$

One can also determine an integral of the following type:

$$\int_{\bar{x}^1}^{\bar{x}^2} \langle\phi(x)\rangle^{(\bar{x})}\, d\bar{x} = \lim_{\Delta\bar{x}\to 0} \Delta\bar{x} \sum_{n=0}^{(\bar{x}_2-\bar{x}_1)/\Delta\bar{x}} (F^{(\bar{x}_1+n\Delta\bar{x})}, \phi(x)F^{(\bar{x}_1+n\Delta\bar{x})}) \quad (5.11)$$

where $\bar{x}$ is the number in the representation in which operator (3.14) is diagonal, i.e., in the representation of "the inner language of the phenomenon"; and also an integral of type (5.9), but depending both on the upper and lower limits

$$\mathscr{I}(x) = \int_{x}^{x+D} \phi(x)\, dx \quad (5.12)$$

where D is an ordinary number. Integrals of type (5.11) and (5.12) may be represented in the purely operational form if we introduce operators of "infinitesimal increase" and "infinitesimal decrease" of the operator.

For example, let $F_0^{(x_0)}$ be such that $x_0 \geqslant 0$.

The operator $P_n{}^+$ for shifting to the right, i.e., an increase by the value $1/2^n$, in this case is of the form

$$\begin{aligned} P_n{}^+ &= a_n{}^+ + a_{n-1}^+ a_n + a_{n-2}^+ a_{n-1} a_n + \cdots \\ &= \sum_{l=0}^{\infty} a_{n-l}^+ a_{n-l+1} \cdots a_n \end{aligned} \quad (5.13)$$

$$P_n{}^+ F_0^{(x_0)} = F_0^{(x_0+1/2^n)}, \qquad x_0 \geqslant 0 \quad (5.14)$$

The inverse and at the same time the Hermitian conjugate operator is

$$P_n = \sum_{l=0}^{\infty} a_{n-l} a_{n-l+1}^+ \cdots a_n{}^+, \qquad P_n{}^+ P_n = 1 \quad (5.15)$$

Here operators

$$a_n = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \tfrac{1}{2}(\sigma_{1n} - i\sigma_{2n}), \qquad a_n{}^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_n \quad (5.16)$$

are operators for voiding and filling the nth cell:

$$\begin{aligned} a_n \varphi_{0n}^1 &= \varphi_{0n}, \qquad & a_n{}^+ \varphi_{0n}^1 &= 0 \\ a_n \varphi_{0n}^0 &= 0, \qquad & a_n{}^+ \varphi_{0n}^0 &= \varphi_{0n}^1 \end{aligned} \quad (5.17)$$

Operators (5.13) and (5.15) act as follows. If cell n is void then only the first term (the operator $a_n{}^+$) will act, filling this cell while the other terms give zero.

If the nth cell is filled but the $(n - 1)$th is void, the second term voids the nth cell and fills the $(n - 1)$th cell, while all the other operators give zero; etc. In all cases x_0 is increased by one and the same value $1/2^n$.

Employing these operators one can write integral (5.12) as

$$\begin{aligned}\mathscr{I}(x) &= \int_x^{x+D} \phi(x)\,dx \\ &= \lim_{n\to\infty} \frac{1}{2^n} [\phi(x) + P_n\phi(x)P_n^+ + P_n^2\phi(x)P_n^{+2} + \cdots + P_n^N\phi(x)P_n^{+N}]\end{aligned} \tag{5.18}$$

where $N = D\cdot 2^n$. In particular if this operator acts on an eigenstate for operator x (i.e., on a "classical" state), then (5.18) transforms into the common classical integral.

If operators P_n and P_n^+ are replaced in (5.18) by operators P_n^+ and $\bar{P}_n^+$ expressed using $\bar{\sigma}_{1n}$ and $\bar{\sigma}_{2n}$ in such a manner that $\bar{\sigma}_{3n}$ is a diagonal operator of type (3.13)

$$\bar{P}_n^+ F^{(\bar{x})} = F^{(\bar{x}+1/2^n)} \tag{5.19}$$

then we get integral (5.11) instead of (5.18).

In the general case when operator (5.18) acts upon the state

$$F^{(\bar{x})} = \sum_{x_0} a(x_0) F_0^{(x_0)}$$

we have

$$\left(\int_x^{x+D} \phi(x)\,dx\right) F^{(\bar{x})} = \lim_{n\to\infty} \frac{1}{2^n} \sum_{x_0} a(x_0) \int_{x_0}^{x_0+D} \phi(y)\,dy F^{(x_0)} \tag{5.20}$$

where the y integral is the usual classical integral. If $\phi(y)$ possesses an antiderivative, $\phi(y) = d\Phi/dy$, then

$$\left(\int_x^{x+D} \phi(x)\,dx\right) F^{(\bar{x})} = \sum_{x_0} a(x_0)[\Phi(x_0 + D) - \Phi(x_0)] F^{(x_0)} \tag{5.21}$$

which coincides with the integral operator definition previously defined:

$$\int_x^{x+D} \phi(x)\,dx = \Phi(x + D) - \Phi(x) \tag{5.22}$$

6. CONCLUSION

The "bone yield" of this paper seems to be formulas (2.11), (5.5), and the result (3.6).

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